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Mixed Equilibria in Games of Strategic Complementarities

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Abstract

The literature on games of strategic complementarities (GSC) has focused on pure strategies. I introduce mixed strategies and show that, when strategy spaces are one-dimensional, the complementarities framework extends to mixed strategies ordered by first-order stochastic dominance. In particular, the mixed extension of a GSC is a GSC, the full set of equilibria is a complete lattice and the extremal equilibria (smallest and largest) are in pure strategies. The framework does not extend when strategy spaces are multi-dimensional. I also update learning results for GSC using stochastic fictitious play.

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1 Introduction

Despite some controversy, many game theorists believe that, in certain circumstances, mixed-strategy equilibria are good predictions of play. Analysis of games of strategic complementarities (GSC) has until now focused on pure-strategy equilibria (see e.g. Topkis (1979), Vives (1990), Milgrom and Roberts (1990) and Milgrom and Shannon (1994)). [1] It is natural to ask if the conclusions in the literature on GSC hold when also mixed-strategy equilibria are considered. In this paper I extend the main results in the seminal papers on GSC to allow for mixed strategies.

I show that, in GSC with one-dimensional strategy spaces, the structure of pure-strategy complementarities extends to mixed strategies when these are ordered by first-order stochastic dominance. Thus the mixed extension of a GSC with one-dimensional strategy spaces is a GSC. It follows that the set of mixed strategy Nash equilibria has a particular order structure (it is a complete lattice) where the extremal equilibria—the largest and smallest equilibria in the sense of first-order stochastic dominance—are in pure-strategies.

In GSC with multidimensional strategy spaces, the complementarities structure does not go through to mixed strategies. The reason for this is technical.

Besides equilibrium predictions, the literature on GSC has analyzed learning processes. Milgrom and Roberts (1990) show that the empirical distribution of play under adaptive learning is, in the limit, bounded by the extremal Nash equilibria. Milgrom and Roberts use a different learning model, and a laxer criterion for convergence, than in the literature on learning mixed-strategy equilibria.

For learning mixed-strategy equilibria Fudenberg and Kreps (1993) propose as convergence requirement that that intended play converges. For example, consider best-response dynamics in "Matching pennies": Start with play being (Heads,Heads), suppose that player 1 wants to match play by 2 and that 2 wants to avoid a match. Then, if be-

[1]This is probably because strategic complementarities guarantee that equilibria in pure strategies exist.

	O	B
O	1,2	0,0
B	0,0	2,1

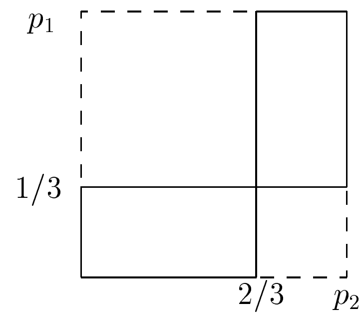

Figure 1: Battle of the Sexes

liefs are that opponents will not change their choices, second period play will be (Heads, Tails). Then third period play will be (Tails, Tails), then (Tails, Heads), then (Heads, Heads) and the cycle starts again. Play will cycle, with each player choosing heads half the time and tails half the time. The empirical distribution of play clearly converges to the mixed strategy equilibrium of matching pennies. The cycle is very simple, though, and it is likely that a real player would discover it and use it to improve her payoff. A player that recognizes the cycle could always extract her maximal payoff in every round of play. In this example the sequence of *intended* play is (Heads, Heads), (Heads, Tails), (Tails, Tails), and so on, which clearly does not converge.

I extend Milgrom and Roberts's results to the learning model and convergence criterion that is standard in the results on learning mixed strategies. I show that it is not possible to refine Milgrom and Roberts's bounds. In particular, one does not obtain convergence to some subset of mixed strategy equilibria.

I shall use the game "Battle of the Sexes" in Figure 1 to illustrate some of the results. Players 1 and 2 choose each simultaneously an element from $\{O, B\}$, the payoffs are specified in the bimatrix to the left. Player 1's best response to 2 playing B is to play B and her best response to 2 playing O is to play O. So, a change by 2 from B to O makes 1 change in the same direction. This is also true for player 2: a change by 1 from B to O makes 2 change in the same direction. Imposing an order on the players' strategies, we can say that O is "larger" than B. Then the best response of each player is increasing in the other player's choice of strategy, this is the crucial property of GSC, and it is easily seen that Battle of the Sexes satisfies the definition of a GSC in e.g. Vives (1990). There are two pure-strategy Nash equilibria of Battle of the Sexes, (O,O) and (B,B); and (O,O) is larger than (B,B).

Now, consider the mixed extension of Battle of the Sexes. That is the game obtained by allowing 1 and 2 to choose probability distributions over $\{O, B\}$ and where payoffs are the expected value of the corresponding pure-strategy payoffs. Let p_i be the probability with which player i selects O. The best responses are shown in Figure 1 on the right. When 2 plays O with probability smaller than 2/3, 1 sets $p_1 = 0$; when p_2 equals 2/3 player 1 is indifferent between O and B, so any choice of p_1 is a best response; when 2 sets p_2 larger than 2/3, 1 will optimally respond by choosing $p_1 = 1$. Hence, player 1's best response is increasing in 2's choice of p_2. The same is true for 2's optimal choice of p_2. This implies that also the mixed extension of Battle of the Sexes has the crucial property of GSC.

There are three Nash equilibria of this game, indicated by the three points where the best-response functions intersect, they are $(0,0), (1/3, 2/3)$ and $(1,1)$. Note that $(0,0)$ is smaller than $(1/3, 2/3)$, which is smaller than $(1,1)$, and that the two extremal equilibria (the smallest and the largest) are in pure strategies. I prove that these features generalize to all GSC with one-dimensional strategy spaces.

Section 2 presents definitions and basic results leading. In Section 3 I show that the mixed extension of a GSC is a GSC, and that the extremal equilibria are in pure strategies. In Section 4 I present the results on the global stability of stochastic fictitious play.

2 Basic Results

2.1 Definitions

A textbook discussion of the concepts defined is in Topkis (1998). A set X with a transitive, reflexive, antisymmetric binary relation $\preceq$ is a ***lattice*** if whenever $x, y \in X$, both $x \wedge y = \inf\{x, y\}$ and $x \vee y = \sup\{x, y\}$ exist in X. It is ***complete*** if for every nonempty subset A of X, $\inf A, \sup A$ exist in X. A nonempty subset A of X is a ***sublattice*** if for all $x, y \in A$, $x \wedge_X y, x \vee_X y \in A$, where $x \wedge_X y$ and $x \vee_X y$ are obtained taking the infimum and supremum as elements of X (as opposed to using the relative order on A). A nonempty subset $A \subset X$ is ***subcomplete*** if $B \subset A$, $B \neq \emptyset$ implies $\inf_X B, \sup_X B \in A$, again taking inf and sup of B as a subset of X.

Let X be a lattice. $A \subset X$ is ***increasing*** if, for all $x \in A$, $y \in X$ and $x \preceq y$ imply $y \in A$. Endow X with a topology, let $\mathcal{P}(X)$ denote the set of Borel probability measures over X. For $\mu, \nu \in \mathcal{P}(X)$, μ is smaller than ν in the ***first-order stochastic dominance order*** (denoted $\mu \leq_{st} \nu$) if, for any increasing measurable set $A \subset X$, $\mu(A) \leq \nu(A)$. It is easy to see that $\mu \leq_{st} \nu$ if and only if $\int_X f d\mu \leq \int_X f d\nu$ for every $f : X \to \mathbf{R}$ that is monotone increasing and integrable.

If X is a lattice and T a partially ordered set. $f : X \to \mathbf{R}$ is ***supermodular*** if, $\forall x, y \in X$, $f(x) + f(y) \leq f(x \wedge y) + f(x \vee y)$; $f : X \times T \to \mathbf{R}$ has ***increasing differences*** in (x, t) if $x \prec x', t \prec t'$, then $f(x', t) - f(x, t) \leq f(x', t') - f(x, t')$.

2.2 Basic Results

This section presents some simple results that are needed in the rest of the paper.

Lemma 1 *If $X \subset \mathbf{R}$ is compact, then $\mathcal{P}(X)$ ordered by first-order stochastic dominance is a complete lattice.*

Proof: Since $X \subset \mathbf{R}$ I shall identify probability measures with their distribution functions. Let $F, G : X \to [0, 1]$ be two distribution functions on X. We know that F is smaller than G in first-order stochastic dominance if and only if $G(x) \leq F(x)$ for all $x \in X$. It is easy to verify that $\leq_{st}$ is a partial order on $\mathcal{P}(X)$.

Define $H : X \to [0, 1]$ by $H(x) = F(x) \wedge G(x)$, it is easy to check that H is a distribution function. I will show that $H = F \vee G$ in the first-order stochastic dominance order. First, H is larger than both F and G. Second, if H' is larger than F and G, then for all x, $H'(x) \leq F(x) \wedge G(x) = H(x)$. Thus H' is also larger than H. These two claims imply that $H = F \vee G$. The argument that $F \wedge G$ exists is similar. This proves that the probability distributions are a lattice under $\leq_{st}$.

To prove that the lattice is complete, first I show that the weak topology on $\mathcal{P}(X)$ is finer than the order-interval topology. [2] For any $x \in X$, let $U_x = [x, \sup X]$. An order interval $[\mu, \nu]$ in $\mathcal{P}(X)$ is then

$$[\mu, \nu] = \bigcap_{\{U_x : x \in X\}} \left(\{p \in \mathcal{P}(X) : \mu(U_x) \leq p(U_x)\} \cap \{p \in \mathcal{P}(X) : p(U_x) \leq \nu(U_x)\} \right).$$

But for all x, $\{p \in \mathcal{P}(X) : \mu(U_x) \leq p(U_x)\}$ and $\{p \in \mathcal{P}(X) : p(U_x) \leq \nu(U_x)\}$ are weakly closed sets (Aliprantis and Border (1999) Theorem 14.6, note that μ and ν are fixed). Then, order intervals are weakly closed and since the order-interval topology is the coarsest topology for which order intervals are closed, the weak is finer than the order-interval topology.

Now, since X is compact, $\mathcal{P}(X)$ is weakly compact. Then, $\mathcal{P}(X)$ is also compact in the order-interval topology because it is coarser than the weak topology. By the Birkhoff-Frink characterization of completeness, then, $\mathcal{P}(X)$ is a complete lattice. ■

Lemma 1 does not generalize to arbitrary sublattices $X \subset \mathbf{R}^n$. The following counterexample is taken from Kamae, Krengel, and O'Brien (1977), let $X = \{0, 1\}^2$ ordered as a subset of $\mathbf{R}^2$. Then $1/2(\delta_{(0,0)} + \delta_{(1,0)})$ and $1/2(\delta_{(0,0)} + \delta_{(0,1)})$ are maximal elements of the set of lower bounds on

$$\left\{ 1/2(\delta_{(0,0)} + \delta_{(1,1)}), 1/2(\delta_{(0,1)} + \delta_{(1,0)}) \right\}.$$

This shows that, if X is any complete lattice that contains two unordered elements, then $\mathcal{P}(X)$ is not a lattice when ordered by first order stochastic dominance.

That $\mathcal{P}(X)$ is not a lattice would create a problem in Section 4, when we study convergence of learning processes because we cannot argue that a monotone sequence converges to the supremum of its range (suprema are not everywhere well defined on partially ordered spaces that are not complete lattices). Fortunately, it is not hard to prove that monotone sequences in $\mathcal{P}(X)$ are convergent, as long as $X \subset \mathbf{R}^n$ is compact.

[2]The order-interval topology on a lattice results from the order intervals as a subbase.

Lemma 2 *Let $X \subset \mathbf{R}^n$ be compact. If $\{\mu_n\}$ is a monotone increasing sequence in $\mathcal{P}(X)$, then it converges weakly to a probability measure μ, and $\mu_n \leq_{st} \mu$ for all n.*

Proof: By Kamae, Krengel, and O'Brien's (1977) Proposition 4 there is a probability space $(\Omega, \mathcal{F}, p)$ and a sequence of random variables $\{Z_n\}$, taking values in X that is increasing a.s. and such that, for all n, Z_n is distributed as μ_n. By compactness of X, there is a random variable Z with $Z_n \uparrow Z$ a.s. Then, $\{\mu_n\}$ is weakly convergent to the distribution μ induced by Z on X and, for any increasing integrable function $f : X \to \mathbf{R}$, $f(Z_n(\omega)) \leq f(Z(\omega))$ a.s for all n, so $\int f d\mu_n = \int f(Z_n)dp \leq \int f(Z)dp = \int f d\mu$. Hence, $\mu_n \leq_{st} \mu$ for all n. ■

Lemma 3 *Let $f : X \subset \mathbf{R} \to \mathbf{R}$ be integrable. Then, $p \mapsto \int_X f(x)dp(x)$ is supermodular on $\mathcal{P}(X)$.*

Lemma 3 is probably known to be true; I include it for completeness, to the best of my knowledge it has not been published. In any case, it is easy to see why Lemma 3 is true: If Y is a lattice that is also a vector space, and where $y + z = y \vee z + y \wedge z$ for all $y, z \in Y$, then any linear function on Y is supermodular. Since integrals are linear in probability distributions, the result would follow. The set of probability distributions is not a vector space, though, so the proof is slightly more involved.

Proof of Lemma 3 Let $p_1, p_2 \in \mathcal{P}(X)$ define $E_1 = \{x \in X : p_2([x, \sup X]) \leq p_1([x, \sup X])\}$ and $E_2 = \{x \in X : p_1([x, \sup X]) < p_2([x, \sup X])\}$. Note that (E_1, E_2) is a measurable partition of X. On subsets of E_1, $p_1 \vee p_2$ coincides with p_1 (see the proof of Lemma 1) while $p_1 \wedge p_2$ coincides with p_2. On subsets of E_2, $p_1 \vee p_2$ coincides with p_2 while $p_1 \wedge p_2$ coincides with p_1. Then, $\int_X f(x)dp_1(x) + \int_X f(x)dp_2(x) =$

$$\int_{E_1} f(x)dp_1(x) + \int_{E_2} f(x)dp_1(x) + \int_{E_1} f(x)dp_2(x) + \int_{E_2} f(x)dp_2(x) =$$

$$\int_{E_1} f(x)dp_1 \vee p_2(x) + \int_{E_2} f(x)dp_1 \wedge p_2(x) + \int_{E_1} f(x)dp_1 \wedge p_2(x) + \int_{E_2} f(x)dp_1 \vee p_2(x)$$

$= \int_X f(x)dp_1 \vee p_2(x) + \int_X f(x)dp_1 \wedge p_2(x)$. ■

Lemma 4 *Let $X_1, X_2, \ldots X_n$ be a collection of subsets of* $\mathbf{R}$ *and* $f : X = \times_{i=1}^n X_i \to \mathbf{R}$ *be integrable. If f has increasing differences in (x_i, x_{-i}) for all i, then*

$$(p_1, \ldots p_n) \mapsto \int_X f(x) d(p_1 \times \ldots p_n)(x) \ : \times_{i=1}^n \mathcal{P}(X_i) \to \mathbf{R}$$

has increasing differences in (p_i, p_{-i}) for all i.

Proof: First note that repeated application of Fubini's theorem implies that $(p_1, \ldots p_n)$ is smaller (componentwise) than $(p'_1, \ldots p'_n)$ if and only if $\int_X g(x) d(p_1 \times \ldots p_n)(x) \leq \int_X g(x) d(p'_1 \times \ldots p'_n)(x)$ for every increasing integrable $g : X \to \mathbf{R}$. In other words, the first-order stochastic dominance order on the set of product measures coincides with the product order.

Fix i and $p_i, p'_i \in \mathcal{P}(X_i)$ where $p_i \leq_{st} p'_i$. For any (componentwise) $x_{-i} \leq x'_{-i}$, $\hat{x}_i \mapsto f(\hat{x}_i, x'_{-i}) - f(\hat{x}_i, x_{-i})$ is increasing. Then,

$$\int_{X_i} f(\hat{x}_i, x'_{-i}) dp_i(\hat{x}_i) - \int_{X_i} f(\hat{x}_i, x_{-i}) dp_i(\hat{x}_i) \leq \int_{X_i} f(\hat{x}_i, x'_{-i}) dp'_i(\hat{x}_i) - \int_{X_i} f(\hat{x}_i, x_{-i}) dp'_i(\hat{x}_i),$$

so that $x_{-i} \mapsto \int_{X_i} f(\hat{x}_i, x_{-i}) dp'_i(\hat{x}_i) - \int_{X_i} f(\hat{x}_i, x_{-i}) dp_i(\hat{x}_i)$ is increasing. Now, if $p_{-i} \leq_{st} p'_{-i}$, then,

$$\int_{X_{-i}} \left\{ \int_{X_i} f(\hat{x}_i, \hat{x}_{-i}) dp'_i(\hat{x}_i) \right\} dp_{-i}(\hat{x}_{-i}) - \int_{X_{-i}} \left\{ \int_{X_i} f(\hat{x}_i, \hat{x}_{-i}) dp_i(\hat{x}_i) \right\} dp_{-i}(\hat{x}_{-i})$$
$$\leq$$
$$\int_{X_{-i}} \left\{ \int_{X_i} f(\hat{x}_i, \hat{x}_{-i}) dp'_i(\hat{x}_i) \right\} dp'_{-i}(\hat{x}_{-i}) - \int_{X_{-i}} \left\{ \int_{X_i} f(\hat{x}_i, \hat{x}_{-i}) dp_i(\hat{x}_i) \right\} dp'_{-i}(\hat{x}_{-i}).$$

Using Fubini, $\int_X f d(p'_i \times p_{-i}) - \int_X f d(p_i \times p_{-i}) \leq \int_X f d(p'_i \times p'_{-i}) - \int_X f d(p_i \times p'_{-i})$. This is just saying that $p_{-i} \mapsto \int_X f d(p'_i \times p_{-i}) - \int_X f d(p_i \times p_{-i})$ is increasing. ■

3 Mixed Strategies in Supermodular Games

The lemmas in Section 2 imply almost immediately that, if mixed strategies are ordered by first-order stochastic dominance, the mixed extension of a GSC with one-dimensional strategy spaces is a GSC—this section's Theorem 6. That the set of mixed equilibria is a complete lattice and that the extremal equilibria are in pure strategies follows as a simple corollary.

When strategy spaces are multidimensional, the set of mixed strategies is not a lattice (see the counterexample in Section 2). This implies that we lack the mathematical structure needed for the current theory of complementarities. We need the lattice property of strategies to make sense of increasing best responses when they are not singleton valued. Multiple best responses are always present when dealing with mixed equilibria and there does not seem to be a simple solution to the requirement that strategy spaces be lattices.

A normal-form game is described by a set of players N together with strategy spaces S_i and payoffs $u_i : S = \times_{i \in N} S_i \to \mathbf{R}$ for all $i \in N$. Let $\Gamma = (N, \{(S_i, u_i) : i \in N\})$ be a normal-form game. Here I shall assume that N is finite. The ***mixed extension*** of Γ is the normal-form game obtained when players $i \in N$ are allowed to choose randomizations $\sigma_i \in \Sigma_i = \mathcal{P}(S_i)$ over the strategies in S_i. The randomizations are assumed to be independent, so a strategy profile σ is a collection $(\sigma_i)_{i \in N}$ that induces a distribution over S by independently mixing the marginals σ_i. Payoffs are $U(\sigma) = \int_S u(s) d\sigma(s)$. Thus the mixed extension of Γ is the normal-form game $(N, \{(\Sigma_i, U_i) : i \in N\})$.

Definition 5 *A normal-form game* $\Gamma = (N, \{(S_i, u_i) : i \in N\})$ *is a **supermodular game** if, for all* $i \in N$,

1. S_i *is a complete lattice;*

2. u_i *is bounded,* $s_i \mapsto u_i(s_i, s_{-i})$ *is supermodular and* $(s_i, s_{-i}) \mapsto u_i(s_i, s_{-i})$ *has increasing differences;*

3. $s_i \mapsto u_i(s_i, s_{-i})$ *is upper semicontinuous and* $s_{-i} \mapsto u_i(s_i, s_{-i})$ *is continuous.*

If, in addition, for all $i \in N$ $S_i \subset \mathbf{R}$ *then it is called a **simple supermodular game**.*

Theorem 6 *If* $(N, \{(S_i, u_i) : i \in N\})$ *is a simple supermodular game then its mixed extension* $(N, \{(\Sigma_i, U_i) : i \in N\})$ *is a supermodular game.*

Proof: Lemmas 1, 3, and 4 almost complete the proof. We just need to note that, for all $i \in \mathbf{N}$, (weak) upper semicontinuity and continuity of $U_i(\sigma_i, \sigma_{-i})$ in σ_i and σ_{-i}, respectively, follow from standard results (see Aliprantis and Border (1999) Theorem 14.5). ■

Corollary 7 *If* $\Gamma = (N, \{(S_i, u_i) : i \in N\})$ *is a simple supermodular game then the set of mixed equilibria of* Γ *is a non-empty complete lattice, and its extremal equilibria (largest and smallest) are in pure strategies.*

Proof: That the set of equilibria is a non-empty complete lattice follows from Zhou's (1994) fixed point theorem. Let $\mathcal{E}$ be the set of pure-strategy equilibria of Γ. Since $\mathcal{E}$ is a complete lattice, $\inf \mathcal{E}, \sup \mathcal{E} \in \mathcal{E}$ and, by Milgrom and Roberts (1990), the set of serially undominated pure strategies is a subset of the order interval $[\inf \mathcal{E}, \sup \mathcal{E}]$. Let σ be a mixed-strategy equilibrium of Γ, since $[\inf \mathcal{E}, \sup \mathcal{E}]$ is a closed set that contains all serially undominated strategies, the support of σ is in $[\inf \mathcal{E}, \sup \mathcal{E}]$. Then, it is easy to show that $\delta_{\inf \mathcal{E}} \leq_{st} \sigma \leq_{st} \delta_{\sup \mathcal{E}}$. Since $\delta_{\inf \mathcal{E}}$ and $\delta_{\sup \mathcal{E}}$ are equilibria, the extremal equilibria, in the sense of first-order stochastic dominance, are in pure strategies. ■

4 Learning Mixed Strategies

The model of learning presented here is similar to the one in Fudenberg and Kreps (1993). Learning takes place through repeated play of a stage game; the stage game is one of incomplete information. Player i's type is $\omega_i \in \Omega_i$, the type space Ω_i is assumed to be a compact topological space. The set of all type profiles is $\Omega = \times_{i \in N} \Omega_i$. In each stage of the repeated game, a type profile $\omega \in \Omega$ is drawn at random, player i is informed of ω_i and chooses a stage-game strategy $s_i \in S_i$. When a strategy profile $s \in S$ is chosen, the payoff to i in the stage game is $u_i(\omega_i)(s)$. Let $\omega_i \mapsto u_i(\omega_i)(s)$ be a continuous function for all $i \in N$ and $s \in S$.

Let p be a probability measure over $\Omega^\infty = \Omega^{\mathbf{N}}$, the space of all sequences of draws (Ω^∞ is endowed with the canonical σ-algebra obtained from the Borel subsets of Ω). I will assume that sequences of type profiles $\omega^\infty \in \Omega^\infty$ are drawn according to p.

The present setup embeds two important special cases: literally mixed strategies and "purified" mixed strategies.

1. (Mixed Strategies) Let $u_i(\omega_i)$ be independent of ω_i. The type spaces represent only randomization devices. In this case, strategies are simply the mixed or correlated strategies that extend the stage game.

2. (Purification) Let $\Omega_i \subset \mathbf{R}^{S_i}$ and $u_i^\delta(\omega_i) = g_i + \delta\omega_i$ for $\delta > 0$ and an integrable $g_i \in \mathbf{R}^S$. This is the setup of Harsanyi's Purification Theorem. Let Γ^δ be the resulting game of incomplete information. Harsanyi's Theorem says that, in generic finite games, for any mixed equilibrium σ of Γ^0 there is a collection (σ_δ) in Σ, where σ_δ is a (pure) equilibrium of Γ^δ, such that $\sigma = \lim_{\delta\to 0}\sigma_\delta$.

The motivation for studying learning mixed strategies in the purification setup is that we need a reason for why players should randomize. The only way that intended play can converge to a mixed equilibrium is by randomized intended play, but it will often be the case that players have no reason to randomize. Recall the matching pennies example in the Introduction: in each round players are not indifferent between different choices, they strictly prefer to play either Heads or Tails. But this is exactly the problem that Harsanyi sought to address, the purification setup captures the intuition that players who are close to indifferent between different choices will base their play on small individual differences.

Fudenberg and Kreps (1993), Benaim and Hirsch (1999), Kaniovski and Young (1995), and Ellison and Fudenberg (1999) study learning of mixed equilibria in the purification setup (see chapter 4 in Fudenberg and Levine (1998) for a discussion). For example, Fudenberg and Kreps (1993) prove that the mixed-strategy equilibrium in Matching Pennies is globally stable.

At each stage a pure strategy profile $s \in S$ results from the players' choices. Histories of play $(s_1, \dots s_t)$ are denoted h_t. If $h_t = (s_1, \dots s_t)$ and $s_{t+1} \in S$, then $h_t s_{t+1}$ will denote

$(s_1, \dots s_t, s_{t+1})$. The set of all histories of length t is $H_t = S^t$ and $H = \cup_{t=0}^{\infty} H_t$ is the set of all histories of finite length, including $H_0 = \emptyset$, the "null history".

Each player i chooses a strategy $\xi_i : \Omega_i \times H \to S_i$ and is endowed with beliefs $\mu_i : H \to \mathcal{P}(S_{-i})$. The interpretation is that, at each time t and history h_t, $\mu_i(h_t) \in \mathcal{P}(S_{-i})$ represents i's assessment of her opponents' play in the $t+1$ stage of the game.

If $\xi = (\xi_i)_{i \in N}$ is a collection of strategies for all players and $\mu = (\mu_i)_{i \in N}$ is a collection of beliefs, then the pair (ξ, μ) is a ***system of behavior and beliefs***. Note that I allow player i to believe that her opponents' play is correlated (for a discussion of the importance of this, see Fudenberg and Kreps (1993)).

Definition 8 *A system of behavior and beliefs* (ξ, μ) *is* ***myopic*** *if for all* $i \in N$, $h_t \in H$ *and* $\omega_i \in \Omega_i$, $\xi_i(\omega_i, h_t) \in argmax_{s_i \in S_i} \int u_i(\omega_i)(s_i, \tilde{s}_{-i}) \mu_i(h_t)(d\tilde{s}_{-i})$

The assumption of myopic behavior is very common in the literature on learning. It is restrictive because it implies that players do not attempt to manipulate the future behavior of their opponents. It is usually justified by assuming that, in each round, players are selected at random from a large population to play the stage game, so the likelihood that two particular players will meet more than once to play the stage game is negligible (see chapter 1 of Fudenberg and Levine (1998) for a discussion).

I need two assumptions on beliefs. The first, called weak asymptotic empiricism, requires that, if beliefs converge, *and* intended play converges, then beliefs must asymptotically resemble actual play (this condition is weaker than asymptotic empiricism in Fudenberg and Kreps (1993)). In other words, if intended play converges, I rule out that beliefs converge to a distribution that is different from actual play. The second, called monotonicity, requires that players increase their beliefs—in the sense of first-order stochastic dominance—after observing larger play. Hopenhayn and Prescott (1992) use this assumption (in Markovian models) basically with the same purpose as here: to guarantee that a sequence of play that is monotone increasing after a finite number of rounds does not cease to be increasing. A simple justification for monotone beliefs is that they are "self enforcing", in the sense that if beliefs are monotone then behavior will be monotone.

Definition 9 *Beliefs are* ***weakly asymptotically empirical*** *if, for all $i \in N$ whenever a sequence of play $\{s_t\}$ is convergent, say $s = \lim s_t$, and the resulting sequence of beliefs $\{\mu_i(h_t)\}$ is convergent then $\mu_i(h_t) = \mu_i((s_1, \ldots s_t)) \to \delta_{s_{-i}}$. Beliefs μ are* ***monotone*** *if, for all $i \in N$, $s_{t-1} \leq s_t$ implies that $\mu_i(h_{t-1}) \leq_{st} \mu_i(h_{t-1} s_t)$ and if $h_t \leq h'_t$ implies $\mu_i(h_t) \leq_{st} \mu_i(h'_t)$.*

Theorems 10 and 11 are the main results on "global convergence" of intended play. Theorem 10 says that myopic rules that respond to monotone beliefs about opponents' play are in the limit bounded by the largest and smallest pure strategy equilibria of the game. The result is an extension to randomized play of Milgrom and Roberts's (1990) results. Theorem 11 says that along any "purifying sequence" limit behavior is bounded by a sequence that converges to pure strategy equilibria of the original game. In this setting, Fudenberg and Kreps (1993) present results on global convergence of intended play for a class of 2X2 games. Theorems 10 and 11 contain a weaker conclusion than global convergence, they only bound the limiting behavior of learning processes; but when equilibrium is unique—like in Fudenberg and Kreps—global convergence is obtained.

Theorem 10 *(Mixed Setup) Let Γ be a simple supermodular game. Let (μ, ξ) be a myopic system of behavior and beliefs with monotone, weakly asymptotically empirical beliefs. The smallest and largest pure equilibria of the stage game Γ, $\underline{e}$ and $\overline{e}$, satisfy, for each realization of $\omega^\infty \in \Omega^\infty$, that $\underline{e} \leq \tilde{\xi} \leq \overline{e}$ and $\delta_{\underline{e}} \leq_{st} \tilde{\mu} \leq_{st} \delta_{\overline{e}}$, for all subsequential limits $\tilde{\mu}$ of $\{\mu(h_t)\}$ and $\tilde{\xi}$ of $\{\xi(\omega_t, h_t)\}$.*

Theorem 11 *(Purification Setup) Let Γ be a simple supermodular game. Let $\{\delta_k\}$ be a sequence in $\mathbf{R}_+$ with $\delta_k \to 0$. For each k let (μ_k, ξ_k) be a myopic system of behavior and beliefs with monotone, weakly asymptotically empirical beliefs. Then there is a subsequence $\{\delta_l\}$ and a sequence of bounds $(\underline{e}_l)$ and $(\overline{e}_l)$ such that, for each realization of*

$\omega^\infty \in \Omega^\infty$, a) For all l, $\underline{e}_l \leq \tilde{\xi}_l \leq \overline{e}_l$ and $\delta_{\underline{e}_l} \leq_{st} \tilde{\mu}_l \leq_{st} \delta_{\overline{e}_l}$ for all subsequential limits $\tilde{\mu}_l$ of $\{\mu_l(h_t)\}$ and $\tilde{\omega_t, \xi_l}$ of $\{\xi_l(h_t)\}$. b) The limits $\underline{e} = \lim \underline{e}_l$, $\overline{e} = \lim \overline{e}_l$ exist and are pure equilibria of the game Γ^0.

Lemma 12 is instrumental in proving the results on "global convergence" in the paper. In the lemma, $\mathcal{E}(\omega)$ denotes the set of pure strategy Nash equilibria of the one shot game obtained by fixing $\omega \in \Omega$ and letting ω be common knowledge.

Lemma 12 *Let Γ be a simple supermodular game. Let (μ, ξ) be a myopic system of behavior and beliefs with monotone weakly asymptotically empirical beliefs. There are $\omega', \omega'' \in \Omega$ and $\underline{e} \in \mathcal{E}(\omega'), \overline{e} \in \mathcal{E}(\omega'')$ such that, for each realization of $\omega^\infty \in \Omega^\infty$, $\underline{e} \leq \tilde{\xi} \leq \overline{e}$ and $\delta_{\underline{e}} \leq_{st} \tilde{\mu} \leq_{st} \delta_{\overline{e}}$ for all subsequential limits $\tilde{\mu}$ of $\{\mu(h_t)\}$ and $\tilde{\xi}$ of $\{\xi(\omega_t, h_t)\}$.*

Proof: For any $\omega \in \Omega$ and $p \in \times_{i\in N}\mathcal{P}(S_{-i})$, by Lemma 4, $(s_i, p_i) \mapsto \int u_i(\omega_i)(s_i, \tilde{s}_{-i})dp_i(\tilde{s}_{-i})$ has increasing differences. By Topkis's Theorem, the set of pure-strategy best responses $\beta^s(\omega, p) = \times_{i\in N}\text{argmax}_{s_i \in S_i} \int u_i(\omega_i)(s_i, \tilde{s}_{-i})dp_i(\tilde{s}_{-i})$ is increasing in the strong set order.

First, consider beliefs and behavior as follows. Let initial assessments be $\mu'_0 = (\delta_{\inf S_{-i}})_{i\in N}$ so that for any $h_1 \in S$ the corresponding beliefs $\mu'_1 = \mu(h_1)$ satisfy $\mu_0 = (\delta_{\inf S_{-i}})_{i\in N} \leq_{st} \mu'_1$. Then, for any $\omega \in \Omega$, $\beta^s(\omega, \mu'_0)$ is smaller than $\beta^s(\omega, \mu'_1)$ in the strong set order. In particular, $\inf\{\beta^s(\omega, \mu'_0) : \omega \in \Omega\} \leq \inf\{\beta^s(\omega, \mu'_1) : \omega \in \Omega\}$. Also, these infima are achieved for some $\omega \in \Omega$ since $\omega \mapsto u_i(\omega)$ is continuous and S_i is a compact subset of $\mathbf{R}$. Now, construct a sequence of play $\{x_t\}$ and probability assessments over opponents' play $\{\mu'_t\}$ recursively by $x_t = \inf\left\{\beta^s(\omega, \mu'_{t-1}) : \omega \in \Omega\right\}$ and $\mu'_t = \mu(h_t)$. Since $x_0 \leq x_1$, $\mu'_0 \leq_{st} \mu'_1$ and the maps $x_t \mapsto \mu(h_{t-1}x_t)$ and $\mu'_t \mapsto \inf\{\beta^s(\omega, \mu'_t) : \omega \in \Omega\}$ are monotone increasing it is clear by induction that the sequences $\{x_t\}$ and $\{\mu'_t\}$ are monotone increasing.

The sequence $\{x_t\}$ is convergent because it is a monotone increasing sequence on a bounded set $S \subset \mathbf{R}^n$. Say that $\underline{e} = \lim x_t$. The sequence $\{\mu'_t\}$ is in $\times_{i\in N}\mathcal{P}(S_{-i})$, it is componentwise monotone increasing. By Lemma 2, $\{\mu'_t\}$ is convergent. Since beliefs are weakly asymptotically empirical, $\delta_{\underline{e}} = \lim \mu'_t$. For each t, there is $\omega'_t \in \Omega$ such that

$x_t \in \beta^s(\omega_t', \mu_t')$. Since Ω is compact we can say, after dropping to a subsequence, that $\{\omega_t\}$ is convergent and set $\omega' = \lim \omega_t'$. Now, $\{(x_t, \mu_t', \omega_t')\}$ is a convergent sequence in the graph of β^s. By upper-hemicontinuity of β^s, $\underline{e} \in \beta^s(\omega', \delta_{\underline{e}})$. This means that $\underline{e}$ is a Nash equilibrium for the stage game obtained when $\omega' \in \Omega$ is drawn, i.e. $\underline{e} \in \mathcal{E}(\omega')$.

Similarly, it is possible to construct a sequence of play $\{y_t\}$ and probability assessments over opponents' play $\{\mu_t''\}$ by setting $\mu_0'' = (\delta_{\sup S_{-i}})_{i \in N}$, $\mu_t'' = \mu(h_t)$ and $y_t = \sup\left\{\beta^s(\omega, \mu_{t-1}'') : \omega \in \Omega\right\}$. Repeating the argument above we obtain a convergent sequence $\{(y_t, \mu_t'', \omega_t'')\}$, say $(\overline{e}, \mu'', \omega'') = \lim_t (y_t, \mu_t'', \omega_t'')$ and $\overline{e} \in \beta^s(\omega'', \delta_{\overline{e}})$ so that $\overline{e} \in \mathcal{E}(\omega'')$.

Fix $\omega^\infty \in \Omega^\infty$. I will show by induction that the sequence of play and probability assessments over opponents' play $\{\mu(h_t), \xi(\omega_t, \mu(h_t))\}$ satisfies $\mu_t' \leq_{st} \mu(h_t) \leq_{st} \mu_t''$ and $x_t \leq \xi(\omega_t, \mu(h_{t-1})) \leq y_t$. First, $\mu_0' \leq_{st} \mu(h_0) \leq_{st} \mu_0''$ by the definitions of μ_0' and μ_0''. By monotonicity of $p \mapsto \beta^s(\tilde{\omega}, p)$, $x_1 = \inf\{\beta^s(\tilde{\omega}, \mu_0') : \tilde{\omega} \in \Omega\} \leq z \leq \sup\{\beta^s(\tilde{\omega}, \mu_0') : \tilde{\omega} \in \Omega\} = y_1$ for all $z \in \beta^s(\tilde{\omega}, \mu_0')$. In particular, since myopic behavior implies that $\xi(\omega_1, \mu(h_0)) \in \beta^s(\omega_1, \mu(h_0))$, $x_1 \leq \xi(\omega_1, \mu(h_0)) \leq y_1$. Second, suppose that $\mu_{t-1}' \leq_{st} \mu(h_{t-1}) \leq_{st} \mu_{t-1}''$ and $x_l \leq \xi(\omega_l, \mu(h_{l-1})) \leq y_l$ for $1 \leq l \leq t-1$. Using monotonicity of $p \mapsto \beta^s(\omega, p)$ again, $\mu_{t-1}' \leq_{st} \mu(h_{t-1}) \leq_{st} \mu_{t-1}''$ implies that $x_t \leq \xi(\omega_t, \mu(h_{t-1})) \leq y_t$. By monotonicity of beliefs,

$$\mu(x_1, \ldots x_t) \leq_{st} \mu\left[\xi(\omega_1, \mu(h_0)), \ldots \xi(\omega_t, \mu(h_{t-1}))\right] \leq_{st} \mu(y_1, \ldots y_t).$$

Then, $\mu_t' = \mu(x_1, \ldots x_t)$ and $\mu_t'' = \mu(y_1, \ldots y_t)$ imply that $\mu_t' \leq_{st} \mu(h_t) \leq_{st} \mu_t''$. This proves that $\mu_t' \leq_{st} \mu(h_{t-1}) \leq_{st} \mu_t''$ and $x_t \leq \xi(\omega_t, \mu(h_{t-1})) \leq y_t$ for all t.

Now, $x_t \to \underline{e}$, $y_t \to \overline{e}$, since $\omega^\infty \in \Omega^\infty$ was arbitrary the first conclusion follows. Also $\mu_t' \to \delta_{\underline{e}}$ and $\mu_t'' \to \delta_{\overline{e}}$ weakly and for every increasing subset A of S_{-i}, $\mu_t'(A) \leq \mu(h_t)(A) \leq \mu_t''(A)$. Then, $\mu_t'(A) \to \delta_{\underline{e}}(A)$ and $\mu_t''(A) \to \delta_{\overline{e}}(A)$ implies that if $\mu(h_{t_k})$ is a subsequence converging to $\tilde{\mu} \in \mathcal{P}(S_{-i})$ then $\delta_{\underline{e}}(A) \leq \tilde{\mu}(A) \leq \delta_{\overline{e}}(A)$. Hence, $\delta_{\underline{e}} \leq_{st} \tilde{\mu} \leq_{st} \delta_{\overline{e}}$ a.s. ■

Proof of Theorem 10 Immediate from Lemma 12. ■

Proof of Theorem 11 Let $\beta' : S \times \Omega \times \mathbf{R}_+ \to S$ be the best response correspondence

$\beta'(s,\omega,\delta) = \times_{i\in N}\text{argmax}_{s_i\in S_i} u_i^\delta(\omega_i)(s_i,s_{-i})$. For each k, Lemma 12 provides $\omega_k', \omega_k'' \in \Omega$, $\underline{e}_k \in \mathcal{E}(\omega_k')$ and $\overline{e}_k \in \mathcal{E}(\omega_k'')$ such that $\underline{e}_k, \overline{e}_k$ satisfy the conditions of part a). The sets S and W are compact so there are convergent subsequences $\{(\underline{e}_l, \overline{e}_l, \omega_l', \omega_l'')\}$. Say $(\underline{e}, \overline{e}, \omega', \omega'') = \lim(\underline{e}_l, \overline{e}_l, \omega_l', \omega_l'')$. By the Maximum Theorem, the correspondence β' is upper hemicontinuous and has thus a closed graph. The sequences $\{(\underline{e}_l, \underline{e}_l, \omega_l', \delta_l)\}$ and $\{(\overline{e}_l, \overline{e}_l, \omega_l'', \delta_l)\}$ are convergent sequences in the graph of of β'. Then, the limits $\underline{e}$ and $\overline{e}$ are fixed points of $s \mapsto \beta'(s,\tilde{\omega},0)$ for $\tilde{\omega} = \omega', \omega''$, respectively. The correspondence $s \mapsto \beta'(s,\tilde{\omega},0)$ is the pure strategy aggregate best response correspondence of game Γ^0. ■

References

Aliprantis, C. D., and K. C. Border (1999): *Infinite Dimensional Analysis.* Springer-Verlag.

Benaim, M., and M. W. Hirsch (1999): "Mixed Equilibria and Dynamical Systems Arising from Fictitious Play in Perturbed Games," *Games and Economic Behavior*, 29(1-2), 36–72.

Ellison, G., and D. Fudenberg (1999): "Learning Purified Mixed Equilibria," Forthcoming in Journal of Economic Theory.

Fudenberg, D., and D. M. Kreps (1993): "Learning Mixed Equilibria," *Games and Economic Behavior*, 5(3), 320–67.

Fudenberg, D., and D. K. Levine (1998): *The theory of learning in games.* MIT Press Series on Economic Learning and Social Evolution, vol. 2. Cambridge and London: MIT Press.

Hopenhayn, H. A., and E. C. Prescott (1992): "Stochastic Monotonicity and Stationary Distributions for Dynamic Economies," *Econometrica*, 60(6), 1387–1406.

Kamae, T., U. Krengel, and G. O'Brien (1977): "Stochastic Inequalities on Partially Ordered Spaces," *Annals of Probability*, 5(6), 899–912.

Kaniovski, Y. M., and H. P. Young (1995): "Learning Dynamics in Games with Stochastic Perturbations," *Games and Economic Behavior*, 11(2), 330–63.

Milgrom, P., and J. Roberts (1990): "Rationalizability, Learning and Equilibrium in Games with Strategic Complementarities," *Econometrica*, 58(6), 1255–1277.

Milgrom, P., and C. Shannon (1994): "Monotone Comparative Statics," *Econometrica*, 62(1), 157–180.

Topkis, D. M. (1979): "Equilibrium Points in Nonzero-Sum n-Person Submodular Games," *SIAM Journal of Control and Optimization*, 17(6), 773–787.

——— (1998): *Supermodularity and Complementarity.* Princeton University Press.

Vives, X. (1990): "Nash Equilibrium with Strategic Complementarities," *Journal of Mathematical Economics*, 19(3), 305–321.

Zhou, L. (1994): "The Set of Nash Equilibria of a Supermodular Game Is a Complete Lattice," *Games and Economic Behavior*, 7(2), 295–300.